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**MULTIPURPOSE TERMINAL INTERCEPTORS:
Game-Theoretic Cost Effectiveness Comparisons
for Mixed Forces**

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ABSTRACT

A simple game-theoretic representation of terminal defense against multicomponent offense has been investigated for two cases:

1. Each offense component must be countered by a "threat-specific" interceptor type. For example, reentry vehicles (RVs) are countered by an ABM interceptor such as SPRINT, high-altitude air-to-surface missiles (ASMs) by a high-altitude missile such as Hercules, and low-altitude ASMs by a low-altitude missile such as HAWK.
2. Some interceptors are multipurpose. For example, suppose SPRINT could intercept RVs and high- and low-altitude ASMs, Hercules could intercept high- and low-altitude ASMs, and HAWK could intercept only low-altitude ASMs.

With single-purpose interceptors there is a (game value) offense allocation that forces the defense to respond as if the offense (unknown to the defense) allocated all of its resources to any one of the offense threat types. The outcome, with this defense response, is completely independent of the actual offense allocation (i.e., it is defense-enforceable). Most reasonable offense allocations require essentially the same defense response, since the allocation optimization is "flat" with respect to offense resources but "spiked" (discontinuous) with respect to defense resources.

With multipurpose interceptors, the same statements are true. But the effective number of interceptors available against any threat type is the sum of the numbers of all interceptor types effective against that threat. The result is that the offense's advantage in deploying more than one component is strongly reduced, or even reversed. Several examples are worked out in this paper.

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I. INTRODUCTION AND SUMMARY

This report is one of a series intended to answer a question posed by Dr. Ronald Easley of ODDR&E:

What advantages might there be in using ABM interceptors (such as SPRINT) jointly as air defense interceptors, compared to mixed ABM/air defense employing threat-specific interceptors?

This report considers a simple game in which an offense deploys several types of threats--for example, ballistic missiles and bomber-delivered air-to-surface missiles (ASMs)--each of which may be countered by a specific type of interceptor--for example, SPRINT ABM interceptors vs RVs and HAWK surface-to-air missiles (SAMs) vs ASMs. Questions answered are: what is the offense's advantage in deploying more than one type of threat if single-purpose interceptors are employed, and how is this offense advantage degraded if some of the interceptors are "multipurpose" in that they can be used against more than one type of threat (for example, if SPRINT can be made effective against ASMs as well as ballistic missiles)?

We have generally assumed that the incremental cost to permit the use of the higher-performance interceptor also in the lesser role is negligible. Historical precedent supports this assumption: in the Hercules system the incremental interceptor costs for anti-tactical-ballistic-missile (ATBM), surface-to-surface, low-altitude, and reduced-dead-zone modes were insignificant.* In the ATBM mode there was a radar cost increment, but this is not really germane to the present analysis. HAWK ATBM experience appears to have been similar.

Even if multipurpose interceptors were more expensive, the present methodology applies. For example, it is shown that multipurpose interceptors are effectively less costly by the sum of the (effective) costs of interceptors with which they are jointly effective. Therefore multipurpose interceptors may still be preferred even if they are more costly than the highest-performance interceptor they replace.

* Mostly minor R & D cost.

Assumptions made throughout this report are:

1. The higher-performance interceptor (SPRINT in the example) costs more than the lower-performance one, and the higher-performance interceptor may be used more flexibly--otherwise the defense would only buy the cheaper interceptor.
2. One interceptor can destroy one threat, and both offense and defense are perfectly reliable. (We assume that aircraft stand off beyond the reach of terminal defenses so that the defense must engage ASMs one-on-one.)
3. The defense is fixed terminal: each interceptor defends only a specific target, and the offense knows how many of each interceptor type are at each target.
4. The offense has the last move in targeting. (In respect to procurement and installation of inventory, this report considers both cases: defense moves last, and offense moves last--with last-mover always assumed to have full knowledge of first-mover's inventory, mix, and deployment.*)
5. There are parallel hierarchies of offense and defense cost: a more expensive offense threat requires a more expensive interceptor.**
6. A "cookie cutter" damage law applies: twice as many bombs do twice as much damage.

For the case of single-purpose or "threat-specific" interceptors of two types, against two types of offense threats, this game has been solved by Matheson.¹ His formulation was that both offense and defense had fixed economic resources, known to each other, and he defined the value of the game as the fraction of the target value destroyed.***

* Since the results are equivalent, the game is equivalent to one of simultaneous moves in allocation.

** A "more expensive" offense component, in this report, means more expensive both per unit and per target value destroyed. It is possible to treat particular cases without this restriction, but the assumption is plausible and allows some useful generalizations.

*** As discussed later, this formulation appears most natural.

As Matheson pointed out, fundamental strategies for the two opponents in this game are:

1. The defense allocates both interceptor types among targets proportionally to the target value,^{*} to deny the offense any preference in targeting.
2. The offense uses only one type of threat against any one target, leaving one of the two interceptor types useless at that target.
3. The offense attacks any one target either with enough resources to exhaust its defense and destroy it, or not at all.

This report modifies and extends Matheson's work in the following ways:

1. Multipurpose interceptors are introduced.
2. The analysis is extended from two-against-two to n-against-n components.
3. More emphasis is placed on one-sided games; in particular, the case where the defense moves last--that is, the offense allocation of resources among threat types is known to the defense before it procures and deploys its mix of defense types.
4. The value function is generally taken as the defense cost (required resources) with fixed offense resources and fixed fractional damage (i.e., a maximum-acceptable-damage limit^{**}), rather than fractional damage with fixed offense and defense resources, or offense cost with fixed fractional damage and fixed defense resources. The three problems are shown to be equivalent.

^{*} Integer effects are neglected. This is equivalent to assuming that both defense and offense forces are large.

^{**} This is equivalent to a cost-exchange ratio from the defensive point of view.

In brief, the results of this analysis are:

In general, when both sides play game-value strategies:

1. In the absence of multipurpose interceptors, adding a component to the offense force (within its total-resource constraint) requires that the defense additionally deploy enough interceptors against that component to counter the possibility that the offense might spend all its resources on that component.
2. Multipurpose interceptors strongly reduce, or even reverse, the advantage to the offense of deploying multiple threat types.

For single-purpose interceptors:

1. The offense never buys a component if by allocating its entire resource to that component, it could not buy enough to exceed the defense's maximum-acceptable-damage limit in the absence of defense. (An implied assumption is that the offense knows the damage limit.) Similarly, the defense never buys interceptors to counter such a threat.*
2. Against an offense that does have more than one component, the defense can hold the damage down to the acceptable-damage limit, even if it were completely ignorant of the allocation of offense resources among components, by adopting a "defense-enforceable" strategy of buying enough interceptors of each type to hold the damage down to this limit even if the offense allocated its entire resources to the corresponding threat type.** (Implied assumptions are that the defense knows how many of each threat type the offense could buy with its total resources, and how much damage each type could cause.) .

* These results were previously presented by Matheson.¹

** In this case the outcome is, in fact, indifferent to the offense resource allocation.

3. The offense can force the defense to this "defense-enforceable" strategy, even in the case when the defense knows the precise offense allocation, i.e. allocating its resources in proportion to the interceptor-to-attacker unit cost ratio. For example, if a SPRINT costs the same as a delivered reentry vehicle (RV) and a HAWK costs half as much as a delivered ASM, the offense can force the defense-enforceable allocation by spending twice as much on RVs as on ASMs.

Matheson's results¹ can be transformed to arrive at similar conclusions, but the observation that the defense can be forced to respond to each additional offense component as though, unknown to the defense, all offense resources were allocated to that component, was not explicit even for two-on-two games; here we also generalize this result to n-on-n games.

For multipurpose interceptors:

1. The defense buys cheaper interceptors to counter a cheaper threat only if the offense, by allocating its entire resources to that threat, could exceed the damage limit against the target system defended by the more expensive interceptor already bought in quantities sufficient to counter the more expensive threat.
2. The defense-enforceable strategy is still to buy enough of each interceptor type to enforce the damage limit even if the offense were able to secretly put its entire resources into any one threat type. But the "effective" number of interceptors of any type is the sum of the numbers of interceptors of each type that can counter that threat. For example, if SPRINT can do the job of HAWK, a target defended by nine SPRINTs and seven HAWKs has effectively nine SPRINTs and sixteen HAWKs.
3. The offense can again force the defense to the above-mentioned defense-enforceable strategy even though the

defense is assumed to know the precise offense threat inventory. It does so this time by allocating its resources in proportion to the "effective" interceptor-to-attacker unit cost ratio, where the "effective" unit cost of an interceptor of any type is reduced by the sum of the "effective" unit costs of all less expensive interceptor types that it can be substituted for.*

* The "effective" increases in numbers and reductions in costs apply to arbitrary as well as game-value offense strategies as long as multipurpose interceptors are not "overstocked" with respect to lesser threats--i.e., as long as the defense must make some response to the presence of the lesser threat. (See Sec. III A.)

II. ANALYSIS (TWO-COMPONENT FORCES WITH SINGLE-PURPOSE INTERCEPTORS)

Defense has last move in procurement, after seeing offense mix. Offense has last move in targeting, after seeing defense mix and deployment.

Defense cost = value function; damage fraction and offense cost fixed.

This portion of the analysis parallels Matheson's¹; both interceptors are single-purpose, e.g., ABM interceptors engage only RVs and SAM interceptors engage only ASMs.

The following definitions are used:^{*}

D_1 Number of defenders (interceptors) of the more expensive type (e.g., ABM interceptors)

D_2 Number of defenders (interceptors) of the less expensive type (e.g., SAM interceptors)

A_1 Number of attackers of the type engaged by type-1 defenders (e.g., RVs)

A_2 Number of attackers of the type engaged by type-2 defenders (e.g., ASMs)

A_{01} Number of type-1 attackers it would take to destroy the entire target system if it were undefended

A_{02} Number of type-2 attackers it would take to destroy the entire target system if it were undefended

C_{D1} Cost of one type-1 defender

C_{D2} Cost of one type-2 defender

C_{A1} Cost of one type-1 attacker

C_{A2} Cost of one type-2 attacker^{**}

^{*}The mnemonic notations D_1 , A_1 , D_2 , A_2 , correspond to Matheson's X_1 , X_2 , X_3 , X_4 , respectively.

^{**}In this notation, the constraints footnoted earlier may be written:
 $C_{A1} > C_{A2}$ and $A_{01} C_{A1} > A_{02} C_{A2}$.

v Maximum-acceptable-damage limit (fraction of target system destroyed)

ϕ Defense cost, $C_{D1}D_1 + C_{D2}D_2$

ψ Offense cost (resource), $C_{A1}A_1 + C_{A2}A_2$

Since the defense allocates both of its resources proportionate to target value (to deny the offense any specific targeting preference), and the offense attacks any given target with only one type of threat, but with sufficient quantity to exhaust the defense and destroy the target:

The fraction of the target value destroyed by a given threat type equals the fraction of the target value attacked by that type, which in turn equals the fraction of the defense's stock of single-purpose interceptors that engage that threat.

Then, if v_1 is the fractional damage due to A_1 and v_2 is the fractional damage due to A_2 :

$$v_1 = \frac{A_1 - v_1 D_1}{A_{01}} \quad \text{or} \quad v_1 = \frac{A_1}{D_1 + A_{01}}$$

and

$$v_2 = \frac{A_2 - v_2 D_2}{A_{02}} \quad \text{or} \quad v_2 = \frac{A_2}{D_2 + A_{02}}$$

$$v = v_1 + v_2 = \frac{A_1}{D_1 + A_{01}} + \frac{A_2}{D_2 + A_{02}}$$

Then we have three simultaneous equations in the allocations A_1 , A_2 , D_1 , D_2 :

Fractional Damage

$$v = \frac{A_1}{D_1 + A_{01}} + \frac{A_2}{D_2 + A_{02}}$$

Defense Cost

$$\phi = C_{D1}D_1 + C_{D2}D_2$$

Offense Cost

$$\psi = C_{A1}A_1 + C_{A2}A_2$$

Any one of the three quantities V , ϕ , ψ may be taken as the value function, to be minimized by one side and maximized by the other; then the other two quantities are considered fixed, and the equations for these other two quantities are treated as equations of constraint. The choice of the fractional damage V as the value function corresponds to Matheson's analysis¹ and is, perhaps the most "natural" choice in that both sides--constrained by fixed resources--play the game of allocating these resources to extremalize the outcome of the exchange. The other formulations, with one or the other of the resources taken as value function, may be objected to on the grounds that they make it appear that the offense and defense have implicitly agreed on an outcome--the fractional damage--but these value functions have the advantage of being equivalent to cost exchange ratios. In the Appendix, it is shown that all six of the games obtained from these three choices of value function and the two choices of who moves first (in procuring and deploying inventory) are equivalent, except for some singularities.

In Secs. III and IV, which follow, the defense cost ϕ is taken as the value function, and the defense is taken as having the last move in allocation; that is, the defense chooses D_1 and D_2 to minimize ϕ for known A_1 and A_2 ; the offense, knowing this optimal defense reaction, chooses A_1 and A_2 to maximize the minimum ϕ .

Lagrange's method² of finding extrema with constraints is adopted here and in the Appendix. Following this method, we first set

$$\frac{\partial \phi}{\partial D_i} + \lambda \frac{\partial V}{\partial D_i} + \mu \frac{\partial \psi}{\partial D_i} = 0, \quad i = 1, 2$$

where λ and μ are undetermined constants. These two equations, together with the equation of constraint

$$V = \frac{A_1}{D_1 + A_{01}} + \frac{A_2}{D_2 + A_{02}}$$

are solved for D_1 and D_2 (and λ); the resulting allocations D_1 and D_2 minimize ϕ for given A_1 , A_2 . Carrying out the differentiations:

$$\left. \begin{aligned} C_{D1} - \lambda \frac{A_1}{(D_1 + A_{01})^2} &= 0 & ; \quad D_1 + A_{01} &= \sqrt{\lambda A_1 / C_{D1}} \\ C_{D2} - \lambda \frac{A_2}{(D_2 + A_{02})^2} &= 0 & ; \quad D_2 + A_{02} &= \sqrt{\lambda A_2 / C_{D2}} \end{aligned} \right\} \quad (1)$$

Substituting these into the equation of constraint:

$$V = \frac{A_1}{\sqrt{\lambda A_1 / C_{D1}}} + \frac{A_2}{\sqrt{\lambda A_2 / C_{D2}}}$$

or, solving for $\sqrt{\lambda}$:

$$\sqrt{\lambda} = \frac{\sqrt{A_1 C_{D1}} + \sqrt{A_2 C_{D2}}}{V} \quad (2)$$

The defense allocations to minimize cost are then found by substituting from Eq. 2 into Eqs. 1:

$$\left. \begin{aligned} D_1 &= \left(\frac{A_1}{V} - A_{01} \right) + \frac{\sqrt{A_1 A_2 (C_{D2} / C_{D1})}}{V} \\ D_2 &= \left(\frac{A_2}{V} - A_{02} \right) + \frac{\sqrt{A_1 A_2 (C_{D1} / C_{D2})}}{V} \end{aligned} \right\} \quad (3)$$

In each of these equations, the first term on the right is the number of interceptors required to limit the damage to V against a pure

attack--e.g., $(A_1/V - A_{01})$ AEM interceptors against A_1 RVs, and $(A_2/V - A_{02})$ SAM interceptors against A_2 ASMs. The second terms represent the increased number of interceptors of each type imposed by the presence of the other offense system, i.e., when requiring that the sum of the damages inflicted by RVs and SAMs does not exceed V .

The minimized defense cost, then, is

$$\text{Min } D_1, D_2 \left\{ \phi(V, \psi \text{ constant}) \right\} = \left(\frac{A_1}{V} - A_{01} \right) C_{D1} + \left(\frac{A_2}{V} - A_{02} \right) C_{D2} + 2 \sqrt{\frac{A_1 A_2 C_{D1} C_{D2}}{V}} \quad (4)$$

The next step is to maximize this cost with respect to the offense allocations A_1 and A_2 , subject to the constraint on offense resources. Proceeding as before:

$$\frac{\partial}{\partial A_j} \left[\text{min } D_1, D_2 \left\{ \phi(V, \psi \text{ constant}) \right\} \right] + \lambda \frac{\partial V}{\partial A_j} + \mu \frac{\partial \psi}{\partial A_j} = 0, \quad j = 1, 2$$

which yield

$$\frac{C_{D1} + \sqrt{A_2/A_1} \sqrt{C_{D1} C_{D2}}}{V} + \mu C_{A1} = 0$$

$$\frac{C_{D2} + \sqrt{A_1/A_2} \sqrt{C_{D1} C_{D2}}}{V} + \mu C_{A2} = 0$$

Multiplying the second equation by C_{A1}/C_{A2} and subtracting from the first to eliminate μ gives a quadratic in A_2/A_1 , with the positive root

$$\frac{A_2}{A_1} = \left(\frac{C_{A1}}{C_{A2}} \right)^2 \quad \frac{C_{D2}}{C_{D1}} \quad \text{or} \quad \frac{C_1 A_1}{C_2 A_2} = \frac{C_{D1}/C_{A1}}{C_{D2}/C_{A2}} \quad (5)$$

That is, the optimum offense allocation is to spend resources on the two attacker types in proportion to the respective interceptor-to-attacker

cost ratios C_{D1}/C_{A1} and C_{D2}/C_{A2} . The allocations A_2 and A_1 may be expressed through the equation of constraint

$$\psi = C_{A1}A_1 + C_{A2}A_2$$

yielding the expressions

$$\left. \begin{aligned} C_{A1}A_1 &= \frac{C_{D1}/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})} \psi \\ C_{A2}A_2 &= \frac{C_{D2}/C_{A2}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})} \psi \end{aligned} \right\} \quad (6)$$

If these values of A_1 and A_2 are substituted into Eq. 4 to find the defense allocation corresponding to the maximized value of the minimized defense cost:

$$D_1 = \frac{\psi}{C_{A1}V} - A_{01}, \quad D_2 = \frac{\psi}{C_{A2}V} - A_{02} \quad (7)$$

and the defense has been forced to expend

$$\phi = \left(\frac{\psi}{C_{A1}V} - A_{01} \right) C_{D1} + \left(\frac{\psi}{C_{A2}V} - A_{02} \right) C_{D2} \quad (8)$$

It is interesting to observe that this game-value defense allocation is exactly the same one that the defense would come up with from a simple assumption: that it knows nothing of the offense allocation and that the offense might spend all its resources on either one of the attacker types. For $(\psi/C_{A1}) - VA_{01}$ is just the total number of type-1 attackers the offense could buy with its total resource, less the number the defense would be willing to let through to destroy a fraction V of the targets; and since only a fraction V of the type-1 defenders could be used, the defense would have to deploy $\frac{1}{V} [(\psi/C_{A1}) - VA_{01}]$ of the type-1 defenders; and similarly for type 2.

Note that in order for D_1 and D_2 to be non-negative, we must have

$$(\psi/C_{A1}) - v_{A01} \geq 0$$

$$(\psi/C_{A2}) - v_{A02} \geq 1$$

That is, the offense must not allocate any resources to an attacker type unless, with his total resources, he could buy enough of that type to exceed the damage limit in the absence of defense. If the offense were to violate this constraint, the defense would simply not allocate any resources to defend against that offense type.

III. GENERALIZATIONS--MULTICOMPONENT FORCES AND MULTIPURPOSE INTERCEPTORS

Defense has last move in procurement and deployment, after seeing offense mix.

Offense has last move in targeting, after seeing defense mix and deployment.

Defense cost = value function; fractional damage and offense cost fixed.

A. MULTICOMPONENT FORCES--SINGLE-PURPOSE INTERCEPTORS

Now consider that there are three types of offense and defense systems, e.g.,

Offense: RVs, high-altitude ASMs, low-altitude ASMs

Defense: ABMs, high-altitude SAMs, low-altitude

Then (with obvious extensions of definitions to D_3 , A_3 , A_{03} , C_{D3} , C_{A3}) let the defense choose

$$D_1 = \frac{\psi}{C_{A1}V} - A_{01} \text{ (if positive--otherwise zero): ABMs vs RVs.}$$

$$D_2 = \frac{\psi}{C_{A2}V} - A_{02} \text{ (if positive--otherwise zero): High-altitude SAMs vs high-altitude ASMs}$$

$$D_3 = \frac{\psi}{C_{A3}V} - A_{03} \text{ (if positive--otherwise zero): Low-altitude SAMs vs low-altitude ASMs}$$

This strategy is, again, defense-enforceable since

$$v_1 + v_2 + v_3 = \frac{A_1}{\psi/C_{A1}V} + \frac{A_2}{\psi/C_{A2}V} + \frac{A_3}{\psi/C_{A3}V} = \frac{C_{A1}A_1 + C_{A2}A_2 + C_{A3}A_3}{\psi} V = V$$

for any A_1 , A_2 , and A_3 .

To minimize the defense cost

$$\phi = C_{D1}D_1 + C_{D2}D_2 + C_{D3}D_3$$

while limiting the damage to

$$V = \frac{A_1}{D_1 + A_{01}} + \frac{A_2}{D_2 + A_{02}} + \frac{A_3}{D_3 + A_{03}} \quad (9)$$

where A_1, A_2, A_3 are given, set

$$\frac{\partial \phi}{\partial D_i} + \lambda \frac{\partial V}{\partial D_i} = 0 \quad ; \quad i = 1, 2, 3$$

which gives

$$\left. \begin{aligned} C_{D1} - \lambda \frac{A_1}{(D_1 + A_{01})^2} &= 0 \quad ; \quad D_1 + A_{01} = \sqrt{\lambda A_1 / C_{D1}} \\ C_{D2} - \lambda \frac{A_2}{(D_2 + A_{02})^2} &= 0 \quad ; \quad D_2 + A_{02} = \sqrt{\lambda A_2 / C_{D2}} \\ C_{D3} - \lambda \frac{A_3}{(D_3 + A_{03})^2} &= 0 \quad ; \quad D_3 + A_{03} = \sqrt{\lambda A_3 / C_{D3}} \end{aligned} \right\} \quad (10)$$

Substituting into Eq. 9:

$$V = \frac{A_1}{\sqrt{\lambda A_1 / C_{D1}}} + \frac{A_2}{\sqrt{\lambda A_2 / C_{D2}}} + \frac{A_3}{\sqrt{\lambda A_3 / C_{D3}}}$$

or, solving for $\sqrt{\lambda}$:

$$\sqrt{\lambda} = \frac{\sqrt{A_1 C_{D1}} + \sqrt{A_2 C_{D2}} + \sqrt{A_3 C_{D3}}}{V} \quad (11)$$

The defense allocations to minimize costs are then found by substituting Eq. 11 into Eqs. 10:

$$D_1 = \left(\frac{A_1}{V} - A_{01} \right) + \frac{\sqrt{A_1 A_2 (C_{D2}/C_{D1})} + \sqrt{A_1 A_3 (C_{D3}/C_{D1})}}{V}$$

$$D_2 = \left(\frac{A_2}{V} - A_{02} \right) + \frac{\sqrt{A_1 A_2 (C_{D1}/C_{D2})} + \sqrt{A_2 A_3 (C_{D3}/C_{D2})}}{V}$$

$$D_3 = \left(\frac{A_3}{V} - A_{03} \right) + \frac{\sqrt{A_1 A_3 (C_{D1}/C_{D3})} + \sqrt{A_2 A_3 (C_{D2}/C_{D3})}}{V}$$

By induction from the two-on-two game (see Eq. 6), take offense allocations:

$$C_{A1} A_1 = \frac{C_{D1}/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) + (C_{D3}/C_{A3})} \psi$$

$$C_{A2} A_2 = \frac{C_{D2}/C_{A2}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) + (C_{D3}/C_{A3})} \psi$$

$$C_{A3} A_3 = \frac{C_{D3}/C_{A3}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) + (C_{D3}/C_{A3})} \psi$$

then

$$\begin{aligned} D_1 &= \left[\frac{C_{D1}/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) + (C_{D3}/C_{A3})} \frac{\psi}{C_{A1} V} - A_{01} \right] \\ &+ \frac{\left(\sqrt{(C_{D1}/C_{A1})^2 (C_{D2}/C_{A2})^2 (C_{D2}/C_{D1})} + \sqrt{(C_{D1}/C_{A1})^2 (C_{D3}/C_{A3})^2 (C_{D3}/C_{D1})} \right) \psi}{[(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) + (C_{D3}/C_{A3})] V} \\ &= \frac{\psi}{C_{A1} V} - A_{01} \end{aligned}$$

Similarly,

$$D_2 = \frac{\psi}{C_{A2} V} - A_{02}$$

$$D_3 = \frac{\psi}{C_{A3}V} - A_{03}$$

The strategies are thus the same as before:

- The defense buys sufficient interceptors of each type to limit the damage to V if the offense elects to spend all of its resources on one (unknown) system.
- The offense allocates its resources in proportion to the interceptor-to-attacker cost ratio.

By induction, for arbitrary numbers of components, n , the optimal defense allocations are:

$$D_1 = (A_1/V - A_{01})$$

$$+ \frac{\sqrt{A_1 A_2 (C_{D2}/C_{D1})} + \sqrt{A_1 A_3 (C_{D3}/C_{D1})} + \dots + \sqrt{A_1 A_n (C_{Dn}/C_{D1})}}{V}$$

$$D_2 = (A_2/V - A_{02})$$

$$+ \frac{\sqrt{A_1 A_2 (C_{D1}/C_{D2})} + \sqrt{A_2 A_3 (C_{D3}/C_{D2})} + \dots + \sqrt{A_2 A_n (C_{Dn}/C_{D2})}}{V}$$

$$D_3 = (A_3/V - A_{03})$$

$$+ \frac{\sqrt{A_1 A_3 (C_{D1}/C_{D3})} + \sqrt{A_2 A_3 (C_{D2}/C_{D3})} + \dots + \sqrt{A_3 A_n (C_{Dn}/C_{D3})}}{V}$$

•
•
•

$$D_n = (A_n/V - A_{0n})$$

$$+ \frac{\sqrt{A_1 A_n (C_{D1}/C_{Dn})} + \sqrt{A_2 A_n (C_{D2}/C_{Dn})} + \dots + \sqrt{A_{n-1} A_n (C_{D(n-1)}/C_{Dn})}}{V}$$

The game-value offense allocations are:

$$c_{A1} A_1 = \frac{C_{D1}/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) + \dots + (C_{Dn}/C_{An})} \psi$$

$$c_{A2} A_2 = \frac{C_{D2}/C_{A2}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) + \dots + (C_{Dn}/C_{An})} \psi$$

$$c_{A3} A_3 = \frac{C_{D3}/C_{A3}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) + \dots + (C_{Dn}/C_{An})} \psi$$

.

.

.

$$c_{An} A_n = \frac{C_{Dn}/C_{An}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) + \dots + (C_{Dn}/C_{An})} \psi$$

forcing the defense-enforceable strategies:

$$D_1 = \frac{\psi}{C_{A1} V} - A_{01}$$

$$D_2 = \frac{\psi}{C_{A2} V} - A_{02}$$

$$D_3 = \frac{\psi}{C_{A3} V} - A_{03}$$

.

.

.

$$D_n = \frac{v}{C_{An} V} - A_{0n}$$

B. MULTIPURPOSE INTERCEPTORS

Return, for the moment, to consideration of two-component forces; e.g., an offense composed of RVs and ASMs opposing a defense of ABMs and SAMs. Consider that the more expensive ABM interceptor can intercept ASMs as well as RVs, while the SAM interceptor can only intercept ASMs.

For single-purpose interceptors, it was clear that the offense would not use more than one type of threat against any one target: it would allocate so as to leave one interceptor type useless at each target. For multipurpose interceptors, it can easily be seen that the offense should, again, use all RVs or all ASMs at any target. The number of RVs needed is

- The number of SPRINTs at the target plus the number of RVs required to destroy the target.

And the number of RVs plus ASMs is

- The number of SPRINTs plus the number of SAMs plus the number of additional objects required to destroy the target.

Since we have assumed that the lesser threat (ASMs in this case) is cheaper both per unit and per unit damage, the offense should not use a mix (e.g., it should use only ASMs in the second case above).

Thus the analysis can proceed on the same assumptions as for single-purpose interceptors. Following the same approach as before:

$$v_1 + v_2 = \frac{A_1}{D_1 + A_{01}} + \frac{A_2}{(D_1 + D_2) + A_{02}}$$

The defense-enforceable strategy is

$$D_1 = \frac{\psi}{C_{A1}V} - A_{01}$$

$$D_1 + D_2 = \frac{\psi}{C_{A2}V} - A_{02}^*$$

and the defense cost is

$$\phi = (C_{D1} - C_{D2}) D_1 + C_{D2} (D_1 + D_2)$$

Then if we define "effective" numbers D'_i and "effective" costs C'_{D_i} by

$$D'_1 = D_1 \quad ; \quad C'_{D1} = C_{D1} - C_{D2}$$

$$D'_2 = D_1 + D_2 \quad ; \quad C'_{D2} = C_{D2}$$

the problem is transformed to an equivalent game with single-purpose interceptors. That is, the defense cost (value function) is identical in form with that of Eq. 8.

*This requires that $\frac{\psi}{C_{A2}V} - A_{02} \geq \frac{\psi}{C_{A1}V} - A_{01}$ for ASMs to be "admissible," i.e., to force the defense to respond by buying SAM interceptors ($D_2 > 0$). This inequality follows from the assumptions that:

1. The more severe threat costs more per unit: $C_{A1} > C_{A2}$

2. It also costs more per target value destroyed:

$$C_{A1}A_{01} > C_{A2}A_{02}$$

3. The offense can buy enough of the more expensive component to exceed the damage limit against the undefended target complex:

$$\frac{\psi}{C_{A1}} > V A_{01}$$

Now consider the case of three interceptor types. Suppose, for example, that:

- The ABM interceptor can intercept all threats.
- The high-altitude interceptor (like e.g., Nike Hercules) can intercept both high-altitude and low-altitude ASMs.
- The low-altitude interceptor (like e.g., HAWK) can intercept only low-altitude ASMs.

Then

$$v_1 + v_2 + v_3 = \frac{A_1}{D_1 + A_{01}} + \frac{A_2}{(D_1 + D_2) + A_{02}} + \frac{A_3}{(D_1 + D_2 + D_3) + A_{03}}$$

giving defense-enforceable strategies

$$D_1 = \frac{\psi}{C_{A1}V} - A_{01}$$

$$D_1 + D_2 = \frac{\psi}{C_{A2}V} - A_{02}$$

$$D_1 + D_2 + D_3 = \frac{\psi}{C_{A3}V} - A_{03}$$

In order for the offense to use high-altitude ASMs (A_2) in the mix it is necessary, as before, that

$$\frac{\psi}{C_{A2}V} - A_{02} \geq \frac{\psi}{C_{A1}V} - A_{01}$$

Similarly, to use low-altitude ASMs (A_3) it is necessary that

$$\frac{\psi}{C_{A3}V} - A_{03} \geq \frac{\psi}{C_{A2}V} - A_{02} \quad \text{and} \quad \frac{\psi}{C_{A2}V} - A_{02} \geq \frac{\psi}{C_{A1}V} - A_{01}$$

or else that

$$\frac{\psi}{C_{A3}V} - A_{03} > \frac{\psi}{C_{A1}V} - A_{01} \quad \text{and} \quad \frac{\psi}{C_{A2}V} - A_{02} \leq \frac{\psi}{C_{A1}V} - A_{01}$$

Then, using only the admissible offense options, it is again possible to transform the game with multipurpose interceptors to an "equivalent" game with single-purpose interceptors.

As an example, for

$$\frac{\psi}{C_{A3}V} - A_{03} > \frac{\psi}{C_{A2}V} - A_{02} > \frac{\psi}{C_{A1}V} - A_{01}$$

the fractional damage would be

$$v_1 + v_2 + v_3 = \frac{A_1}{D_1 + A_{01}} + \frac{A_2}{(D_1 + D_2) + A_{02}} + \frac{A_3}{(D_1 + D_2 + D_3) + A_{03}}$$

and the defense cost

$$\phi = [C_{D1} - (C_{D2} - C_{D3}) - C_{D3}] (D_1) + (C_{D2} - C_{D3}) (D_1 + D_2) + C_{D3} (D_1 + D_2 + D_3)$$

Then the substitutions

$$D_1' = D_1 \quad C_{D1}' = C_{D1} - C_{D2}$$

$$D_2' = D_1 + D_2 \quad C_{D2}' = C_{D2} - C_{D3}$$

$$D_3' = D_1 + D_2 + D_3 \quad C_{D3}' = C_{D3}$$

transform the problem to

$$v_1 + v_2 + v_3 = \frac{A_1}{D_1' + A_{01}} + \frac{A_2}{D_2' + A_{02}} + \frac{A_3}{D_3' + A_{03}}$$

$$\phi = C_{D1}' D_1' + C_{D2}' D_2' + C_{D3}' D_3'$$

i.e., to an equivalent game with single-purpose interceptors.

Optimal defense strategies against arbitrary a priori threat distributions (compare with Sec. II B) are

$$D'_1 = \left(\frac{A_1}{V} - A_{01} \right) + \frac{\sqrt{A_1 A_2 (C'_{D2} / C'_{D1})} + \sqrt{A_1 A_3 (C'_{D3} / C'_{D1})}}{V}$$

$$D'_2 = \left(\frac{A_2}{V} - A_{02} \right) + \frac{\sqrt{A_1 A_2 (C'_{D1} / C'_{D2})} + \sqrt{A_1 A_3 (C'_{D3} / C'_{D2})}}{V}$$

$$D'_3 = \left(\frac{A_3}{V} - A_{03} \right) + \frac{\sqrt{A_1 A_3 (C'_{D1} / C'_{D3})} + \sqrt{A_2 A_3 (C'_{D2} / C'_{D3})}}{V}$$

The game-value offense allocations are:

$$C_{A1} A_1 = \frac{C'_{D1} / C_{A1}}{(C'_{D1} / C_{A1}) + (C'_{D2} / C_{A2}) + (C'_{D3} / C_{A3})} \psi$$

$$C_{A2} A_2 = \frac{C'_{D2} / C_{A2}}{(C'_{D1} / C_{A1}) + (C'_{D2} / C_{A2}) + (C'_{D3} / C_{A3})} \psi$$

$$C_{A3} A_3 = \frac{C'_{D3} / C_{A3}}{(C'_{D1} / C_{A1}) + (C'_{D2} / C_{A2}) + (C'_{D3} / C_{A3})} \psi$$

Forcing the defense response:

$$D'_1 = \frac{\psi}{C_{A1} V} - A_{01}; D'_2 = \frac{\psi}{C_{A2} V} - A_{02}; D'_3 = \frac{\psi}{C_{A3} V} - A_{03}$$

Then:

- The defense, taking account of the multipurpose capabilities of its interceptors, buys sufficient interceptors of each type to limit the damage to V if the offense elects to spend all of its resources on any one system.
- The offense allocates its admissible resources in proportion to the effective interceptor-to-attacker cost ratio,

where the effective interceptor cost is reduced by the effective costs of the interceptors with which it is jointly effective.

Returning to the hypothetical example and calling SPRINT the ABM interceptor, Hercules the high-altitude interceptor, and HAWK the low-altitude interceptor: the effect of having SPRINT capability against ABMs and high- and low-altitude ASMs, Hercules capability against high- and low-altitude ASMs,^{*} and HAWK capability only against low-altitude ASMs is to:

- Reduce the effective SPRINT interceptor unit cost by the sum of the effective Hercules and HAWK unit costs.^{**}
Reduce the effective Hercules interceptor unit cost by the HAWK unit cost.
- Increase the "effective" stock of interceptors against low-altitude ASMs to the sum of the SPRINT, Hercules, and HAWK inventories. Increase the "effective" stock of interceptors against high-altitude ASMs to the sum of the SPRINT and Hercules inventories.

Note that these effects apply not only to game-value strategies but for all offense distributions such that

$$D_3' \geq D_2' \geq D_1'$$

i.e., as long as multipurpose interceptors are not "overstocked" with respect to the lesser threats.

* Provided that the ability to engage low-altitude ASMs could be obtained with negligible cost increase for the Hercules system.

** Since the effective Hercules unit cost is the Hercules cost less the HAWK cost, the effective SPRINT cost is just the SPRINT cost less the actual Hercules cost.

IV. EXAMPLES--TWO-COMPONENT FORCES

Figures 1 and 2 show examples of minimized defense costs vs offense force allocation with SAM-to-ABM interceptor cost ratio, C_{D2}/C_{D1} , as a parameter. Throughout we have taken a damage limit of 25% and have assumed that 500 attackers of either type (RV or ASM) destroy the undefended target system.

In Fig. 1, ASM and RV costs are equal. With 7500 multipurpose interceptors the defense is enforceable against all possible offense distributions. For C_{D2} equal to C_{D1} an all-ABM defense (with multipurpose interceptors) is optimal for all offense distributions. For C_{D2}/C_{D1} equal to one-half, the all-ABM defense is optimal if RVs make up 50 percent or more of the force (an optimal, or game-value, offense strategy) while for C_{D2}/C_{D1} equal to one-quarter the all-ABM defense is optimal if RVs make up 75 percent or more of the force (again an optimal, or game-value, attack strategy). If the offense used all ASMs the defense would buy all SAM interceptors except when multipurpose interceptors cost the same ($C_{D2} = C_{D1}$) in which case the defense would be indifferent.

For C_{D2} equal to C_{D1} , the defense using single-purpose interceptors is twice as costly against an optimized offense and nearly twice as costly against most possible offense mixes.

For C_{D2}/C_{D1} equal to one-half, the defense using single-purpose interceptors is fifty percent more costly against an optimized offense and nearly that much more costly against most possible offense mixes.

In a similar fashion, for C_{D2}/C_{D1} equal to one-quarter, the single-purpose defense is generally about twenty-five percent more costly.

The general advantage of multipurpose interceptors over single-purpose interceptors is quite apparent.

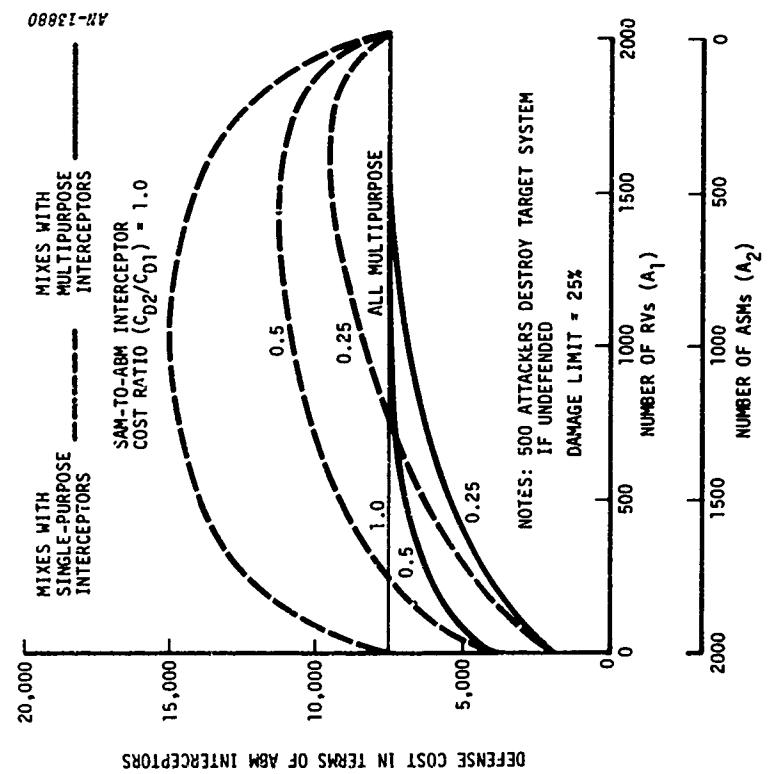
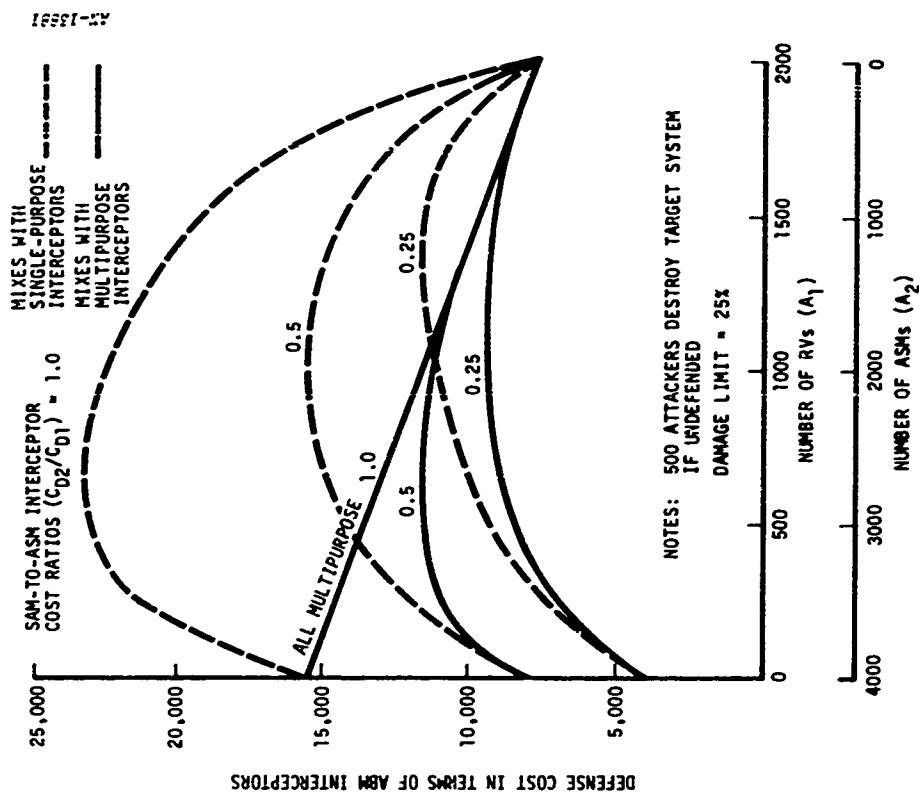


Figure 1. Minimized Defense Cost if RV Cost = ASM Cost ($C_{A1}/C_{A2} = 1.0$)

Figure 2. Minimized Defense Cost if RV Cost = Twice ASM Cost ($C_{A1}/C_{A2} = 2.0$)

In Fig. 2, ASMs are half as costly as RVs. Conclusions are similar except that optimized attacks favor a mix with ASMs, the cost advantage of the multipurpose defense is slightly reduced (especially for large C_{D2}/C_{D1}), and the defense shifts to all (universal) ABMs at relatively higher offense concentrations of RVs in the attack mix.

V. EXAMPLES--MULTICOMPONENT FORCES

In this section we consider an example based on the following hypothetical threat spectrum:

- A_1 RVs with penetration aids
- A_2 simple RVs
- A_3 high-altitude ASMs
- A_4 low-altitude ASMs

All objects have the same yield ($A_{01} = A_{02} = A_{03} = A_{04} = 500$) and all except A_1 have the same unit costs ($C_{A2} = C_{A3} = C_{A4} = 1$). A_1 has double the unit cost ($C_{A1} = 2$), which is equivalent to charging a fifty percent weight penalty for penetration aids (unspecified). The offense resource equals 2000 units. The damage limit, V , is again taken as 0.25.

The defense force is made up of:

- D_1 ABM interceptors
- D_2 SAM interceptors
- D_3 low-altitude SAM interceptors

Throughout we consider that only the ABM interceptor can intercept RVs with penetration aids and that the low-altitude SAM can intercept only low-altitude ASMs. Furthermore, we assume that the ABM-to-SAM-to-low-altitude-SAM cost ratios are $C_{D1}/C_{D2} = C_{D2}/C_{D3} = 2$. Then we investigate the effects of multipurpose ABM and SAM interceptors with the following combinations of capabilities:

- ABM
 - RVs only
 - RVs and low-altitude ASMs

RVs and high-altitude ASMs	
RVs and low- and high-altitude ASMs	
SAM	
High-altitude ASMs only	
High- and low-altitude ASMs	
High-altitude ASMs and simple RVs	
High- and low-altitude ASMs and simple RVs	

Table 1 shows game values (measured in units of ABM interceptors) and game-value strategies for the sixteen possible interceptor combinations. In some cases there are two offense strategies which force the defense-enforceable response (corresponding to cases where the offense has just the threshold value of an admissible component),* while in others there is a range of offense distributions over several components since these components (or any combination thereof) appear equivalent to certain combinations of multipurpose interceptors. This latter effect is particularly noticeable where both ABM and SAM interceptors have maximal multipurpose capabilities (toward the lower right corner of Table 1). In such cases the (optimal) defense becomes quite insensitive to the offense composition. Note, by the way, that the offense buys RVs with penetration aids only if SAM interceptors can intercept simple RVs.

A defense employing two multipurpose interceptors can achieve considerable cost savings (over two-to-one in terms of game value) in comparison with defenses using single-purpose interceptors; and it is much less sensitive to knowledge of the offense composition.

Figure 3 shows the defense cost (game value) vs the number of low-altitude ASMs with 333 RVs with penetration aids (the extreme lower right box of Table 1). For fewer than 667 low-altitude ASMs, the defense cost

*These are quite singular, i.e., any other nonzero value of the offense component yields a lesser game value against a responsive defense.

TABLE 1

GAME VALUES AND STRATEGIES AS FUNCTIONS OF THE DEGREE
MULTIPURPOSE INTERCEPTOR CAPABILITY

RVs ONLY				RVs + Low Alt., ASMs				RVs + High Alt., ASMs				RVs + High Alt., ASMs			
GAME VALUE = 11250 ASMs		11250		9175		9175		7100		7100		7100		7100	
$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$
$D_2 = 7500$	$A_2 = 1163$	$D_2 = 7500$	$A_2 = 1000$	$D_2 = 7500$	$A_2 = 1113$	$D_2 = 7500$	$A_2 = 800$	$D_2 = 7500$	$A_2 = 800$	$D_2 = 7500$	$A_2 = 800$	$D_2 = 7500$	$A_2 = 800$	$D_2 = 7500$	$A_2 = 800$
$D_3 = 7500$	$A_3 = 571$	$D_3 = 7500$	$A_3 = 667$	$D_3 = 7500$	$A_3 = 667$	$D_3 = 7500$	$A_3 = 667$	$D_3 = 7500$	$A_3 = 667$	$D_3 = 7500$	$A_3 = 667$	$D_3 = 7500$	$A_3 = 667$	$D_3 = 7500$	$A_3 = 667$
$D_4 = 0$	$A_4 = 286$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$
GAME VALUE = 11250 ASMs		11250		9175		9175		7100		7100		7100		7100	
$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$	$D_1 = 7500$	$A_1 = 0$
$D_2 = 7500$	$A_2 = 1333$	$D_2 = 7500$	$A_2 = 1113$	$D_2 = 7500$	$A_2 = 1113$	$D_2 = 7500$	$A_2 = 1000$	$D_2 = 7500$	$A_2 = 1000$	$D_2 = 7500$	$A_2 = 1000$	$D_2 = 7500$	$A_2 = 1000$	$D_2 = 7500$	$A_2 = 1000$
$D_3 = 0$	$A_3 = 333$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$
GAME VALUE = 11250 ASMs		11250		9175		9175		7100		7100		7100		7100	
$D_1 = 3500$	$A_1 = 600$	$D_1 = 3500$	$A_1 = 333$	$D_1 = 3500$	$A_1 = 333$	$D_1 = 3500$	$A_1 = 333$	$D_1 = 3500$	$A_1 = 333$	$D_1 = 3500$	$A_1 = 333$	$D_1 = 3500$	$A_1 = 333$	$D_1 = 3500$	$A_1 = 333$
$D_2 = 7500$	$A_2 = 800$	$D_2 = 7500$	$A_2 = 800$	$D_2 = 7500$	$A_2 = 800$	$D_2 = 7500$	$A_2 = 800$	$D_2 = 4000$	$A_2 = 4000$	$D_2 = 4000$	$A_2 = 4000$	$D_2 = 4000$	$A_2 = 4000$	$D_2 = 4000$	$A_2 = 4000$
$D_3 = 7500$	$A_3 = 400$	$D_3 = 7500$	$A_3 = 400$	$D_3 = 4000$	$A_3 = 400$	$D_3 = 4000$	$A_3 = 400$	$D_3 = 7500$	$A_3 = 7500$	$D_3 = 7500$	$A_3 = 7500$	$D_3 = 7500$	$A_3 = 7500$	$D_3 = 7500$	$A_3 = 7500$
$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$	$D_4 = 0$	$A_4 = 0$
GAME VALUE = 11250 ASMs		9175		7100		7100		6175		6175		6175		6175	
$D_1 = 3500$	$A_1 = 500$	$D_1 = 500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$
$D_2 = 7500$	$A_2 = 0$	$D_2 = 0$	$A_2 = 0$	$D_2 = 7500$	$A_2 = 0$	$D_2 = 7500$	$A_2 = 0$	$D_2 = 4000$	$A_2 = 4000$	$D_2 = 4000$	$A_2 = 4000$	$D_2 = 4000$	$A_2 = 4000$	$D_2 = 4000$	$A_2 = 4000$
$D_3 = 0$	$A_3 = 500$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$
GAME VALUE = 11250 ASMs		7250		7250		6175		6175		6175		6175		6175	
$D_1 = 3500$	$A_1 = 500$	$D_1 = 500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$	$D_1 = 3500$	$A_1 = 500$
$D_2 = 7500$	$A_2 = 500$	$D_2 = 500$	$A_2 = 500$	$D_2 = 7500$	$A_2 = 500$	$D_2 = 7500$	$A_2 = 500$	$D_2 = 4000$	$A_2 = 4000$	$D_2 = 4000$	$A_2 = 4000$	$D_2 = 4000$	$A_2 = 4000$	$D_2 = 4000$	$A_2 = 4000$
$D_3 = 0$	$A_3 = 500$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$	$D_3 = 0$	$A_3 = 0$
GAME VALUE = 11250 ASMs		7250		6175		6175		6175		6175		6175		6175	

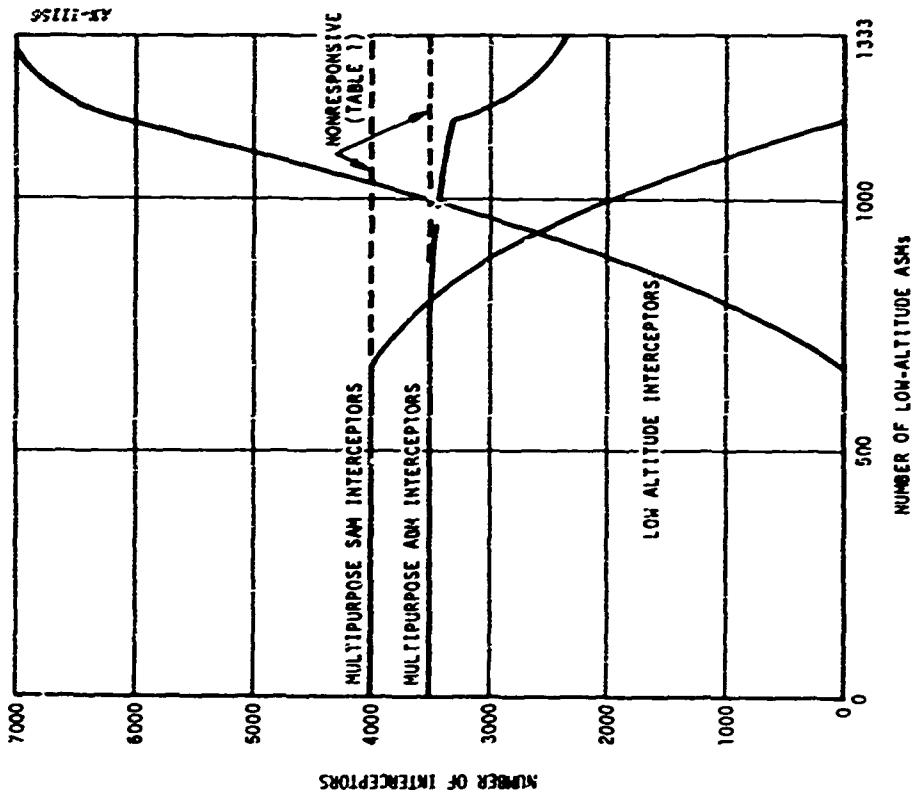


Figure 4. Responsive Interceptor Inventories for the Case Shown in Fig. 3.

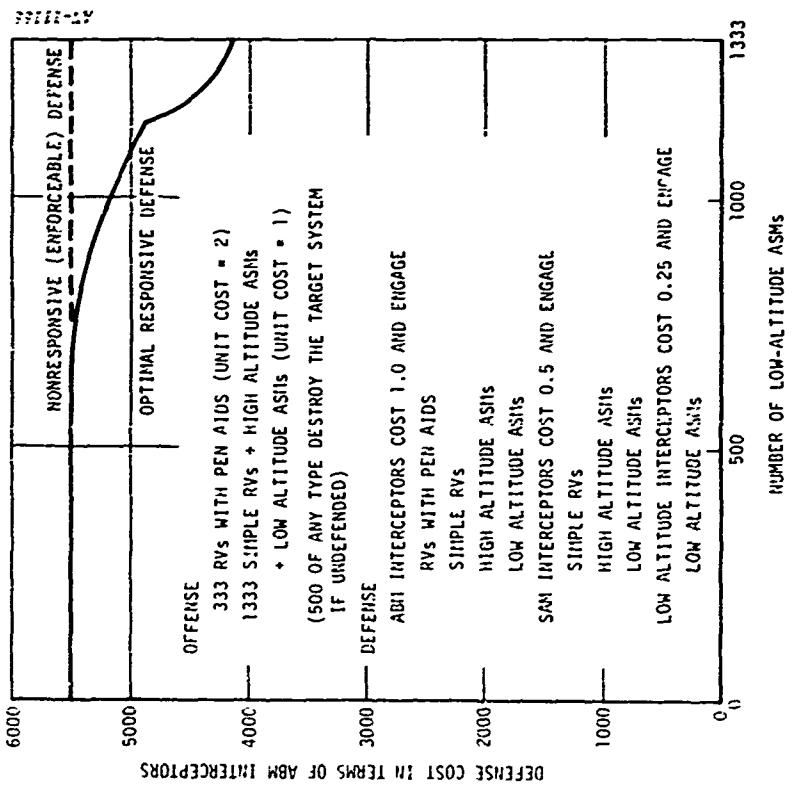


Figure 3. Sensitivity of Defense with Multi-purpose Interceptors to Threat Mix

(and optimal allocation) is totally indifferent to the distribution between simple RVs, high-altitude ASMs, and low-altitude ASMs. For more than 667 low-altitude ASMs, the optimal defense is only slightly less costly than the enforceable defense even when low-altitude ASMs completely replace simple RVs and high-altitude ASMs. As seen from Fig. 4, in order for the defense to gain the small cost advantage against off-optimal offense strategies it must respond drastically to the offense mix, thus incurring considerable sensitivity to knowledge of the offense composition.

APPENDIX
THE THREE* VERSIONS OF THE TWO-SIDED GAME

As mentioned in the text, any of the three quantities V , ϕ , ψ , can be taken as value function with the other two fixed; and either offense or defense may be taken as having the last move in allocation. In this Appendix, Lagrange's method is used to carry out the six operations

$$\text{Max}_{A_1, A_2} \quad \text{Min}_{D_1, D_2} \quad \{V(\phi, \psi \text{ constant})\}$$

$$\text{Min}_{D_1, D_2} \quad \text{Max}_{A_1, A_2} \quad \{V(\phi, \psi \text{ constant})\}$$

$$\text{Max}_{A_1, A_2} \quad \text{Min}_{D_1, D_2} \quad \{\phi(V, \psi \text{ constant})\}^{**}$$

$$\text{Min}_{D_1, D_2} \quad \text{Max}_{A_1, A_2} \quad \{\phi(V, \psi \text{ constant})\}$$

$$\text{Min}_{A_1, A_2} \quad \text{Max}_{D_1, D_2} \quad \{\psi(V, \phi \text{ constant})\}$$

$$\text{Max}_{D_1, D_2} \quad \text{Min}_{A_1, A_2} \quad \{\psi(V, \phi \text{ constant})\}$$

Two types of single-purpose interceptors are assumed throughout. As the text shows, the equations can be applied to multipurpose interceptors by a change of variables and to more than two types by simple extensions. The results are summarized in Table 2, at the end of the Appendix.

* For game-value strategies these six cases represent three zero sum games.

** The case worked out in the text.

As in the text, we start out from the three equations for V , ϕ , and ψ :

Fractional Damage

$$V = \frac{A_1}{D_1 + A_{01}} + \frac{A_2}{D_2 + A_{02}}$$

Defense Cost

$$\phi = C_{D1} D_1 + C_{D2} D_2$$

Offense Cost

$$\psi = C_{A1} A_1 + C_{A2} A_2$$

Since A_{01} and A_{02} are constants we can, without loss of generality, change variables to:

$$D_1' = D_1 + A_{01}$$

$$D_2' = D_2 + A_{02}$$

$$\phi' = C_{D1} D_1' + C_{D2} D_2'$$

The equations become:

Fractional Damage

$$V = \frac{A_1}{D_1'} + \frac{A_2}{D_2'}$$

Equivalent Defense Cost ^{*}

$$\phi' = C_{D1} D_1' + C_{D2} D_2'$$

Offense Cost

$$\psi = C_{A1} A_1 + C_{A2} A_2$$

In the following we suppress the prime notation for convenience. We use Lagrange's method of undetermined multipliers² (λ and μ) throughout.

A. FRACTIONAL DAMAGE AS VALUE FUNCTION
WITH RESOURCES CONSTRAINED

1. $\frac{\text{Max}_{A_1, A_2} \text{Min}_{D_1, D_2} \{V(\phi, \psi \text{ constant})\}}{}$

$$\frac{\partial V}{\partial D_i} + \lambda \frac{\partial \phi}{\partial D_i} + \mu \frac{\partial \psi}{\partial D_i} = 0 ; i = 1, 2$$

$$\frac{-A_2}{D_1^2} + \lambda C_{D1} = 0 ; D_1 = \sqrt{A_1 / \lambda C_{D1}}$$

$$\frac{-A_2}{D_2^2} + \lambda C_{D2} = 0 ; D_2 = \sqrt{A_2 / \lambda C_{D2}}$$

$$\phi = C_{D1} D_1 + C_{D2} D_2$$

These simultaneous equations solve to:

$$D_1 = \frac{\phi}{\sqrt{C_{D1} C_{D2} (A_2 / A_1)} + C_{D1}}$$

^{*}I.e., the "equivalent" defense resource is greater than the defense cost by the amount $C_{D1} A_{01} + C_{D2} A_{02}$.

$$D_2 = \frac{\phi}{\sqrt{C_{D1}C_{D2}(A_1/A_2)} + C_{D2}}$$

$$\text{Min}_{D_1, D_2} \{V(\phi, \psi \text{ constant})\} = \frac{\sqrt{C_{D1}C_{D2}A_1A_2} + C_{D1}A_1 + \sqrt{C_{D1}C_{D2}A_1A_2} + C_{D2}A_2}{\phi}$$

$$= \frac{(\sqrt{C_{D1}A_1} + \sqrt{C_{D2}A_2})^2}{\phi}$$

$$\frac{\partial}{\partial A_j} [\text{Min}_{D_1, D_2} \{V(\phi, \psi \text{ constant})\}] + \lambda \frac{\partial \phi}{\partial A_j} + \mu \frac{\partial \psi}{\partial A_j} = 0 ; i = 1, 2$$

$$\sqrt{\frac{C_{D1}C_{D2}(A_2/A_1)}{\phi}} + C_{D1} + \mu C_{A1} = 0$$

$$\sqrt{\frac{C_{D1}C_{D2}(A_1/A_2)}{\phi}} + C_{D2} + \mu C_{A2} = 0$$

These simultaneous equations yield

$$\frac{A_2}{A_1} = \left(\frac{C_{A1}}{C_{A2}} \right)^2 \frac{C_{D2}}{C_{D1}}$$

Substituting into

$$\psi = C_{A1}A_1 + C_{A2}A_2$$

yields

$$C_{A1}A_1 = \frac{C_{D1}/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})} \psi$$

$$C_{A2}A_2 = \frac{C_{D2}/C_{A2}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})} \psi$$

$$\frac{\text{Max}_{A_1, A_2} \text{Min}_{D_1, D_2} \{V(\phi, \psi \text{ constant})\}}{\phi} = \frac{\psi}{\phi} \left(\frac{C_{D1}}{C_{A1}} + \frac{C_{D2}}{C_{A2}} \right)$$

and the Max Min allocations of D_1 and D_2 are:

$$D_1 = \frac{\phi/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}$$

$$D_2 = \frac{\phi/C_{A2}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}$$

$$2. \frac{\text{Min}_{D_1, D_2} \text{Max}_{A_1, A_2} \{V(\phi, \psi \text{ constant})\}}{\phi/C_{A1} + \mu C_{A2}}$$

$$\frac{\partial V}{\partial A_j} + \frac{\partial \phi}{\partial A_j} + \mu \frac{\partial \psi}{\partial A_j} = 0; \quad j = 1, 2$$

$$\frac{1}{D_1} + \mu C_{A1} = 0$$

$$\frac{1}{D_2} + \mu C_{A2} = 0$$

$$\phi = C_{D1}D_1 + C_{D2}D_2$$

This set of equations only has a solution for

$$D_1 = \frac{\phi/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})} ; \quad D_2 = \frac{\phi/C_{A2}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}$$

$$\text{For } D_1 < \frac{\phi/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}; \quad D_2 > \frac{\phi/C_{A2}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}$$

(because if D_1 decreases, D_2 must increase to satisfy the resource constraint).

and

$$\frac{\partial V}{\partial A_1} + \frac{\partial \phi}{\partial A_1} + \mu \frac{\partial \psi}{\partial A_1} > 0$$

while

$$\frac{\partial V}{\partial A_2} + \frac{\partial \phi}{\partial A_2} + \mu \frac{\partial \psi}{\partial A_2} < 0$$

then the offense buys all A_1 and

$$V = \frac{\psi/C_{A1}}{D_1}$$

If

$$D_1 \geq \frac{\phi/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}$$

the offense buys all A_2 and

$$V = \frac{\psi \left(\frac{C_{D2}}{C_{A2}} \right)}{\phi - C_{D1} D_1}$$

Figure 5 shows an example of maximized fractional damage vs defense allocation for

$$\phi' = 8000$$

$$C_{D1} = 2$$

$$C_{D2} = 1$$

$$\psi = 2000$$

$$C_{A1} = 2$$

$$C_{A2} = 1$$

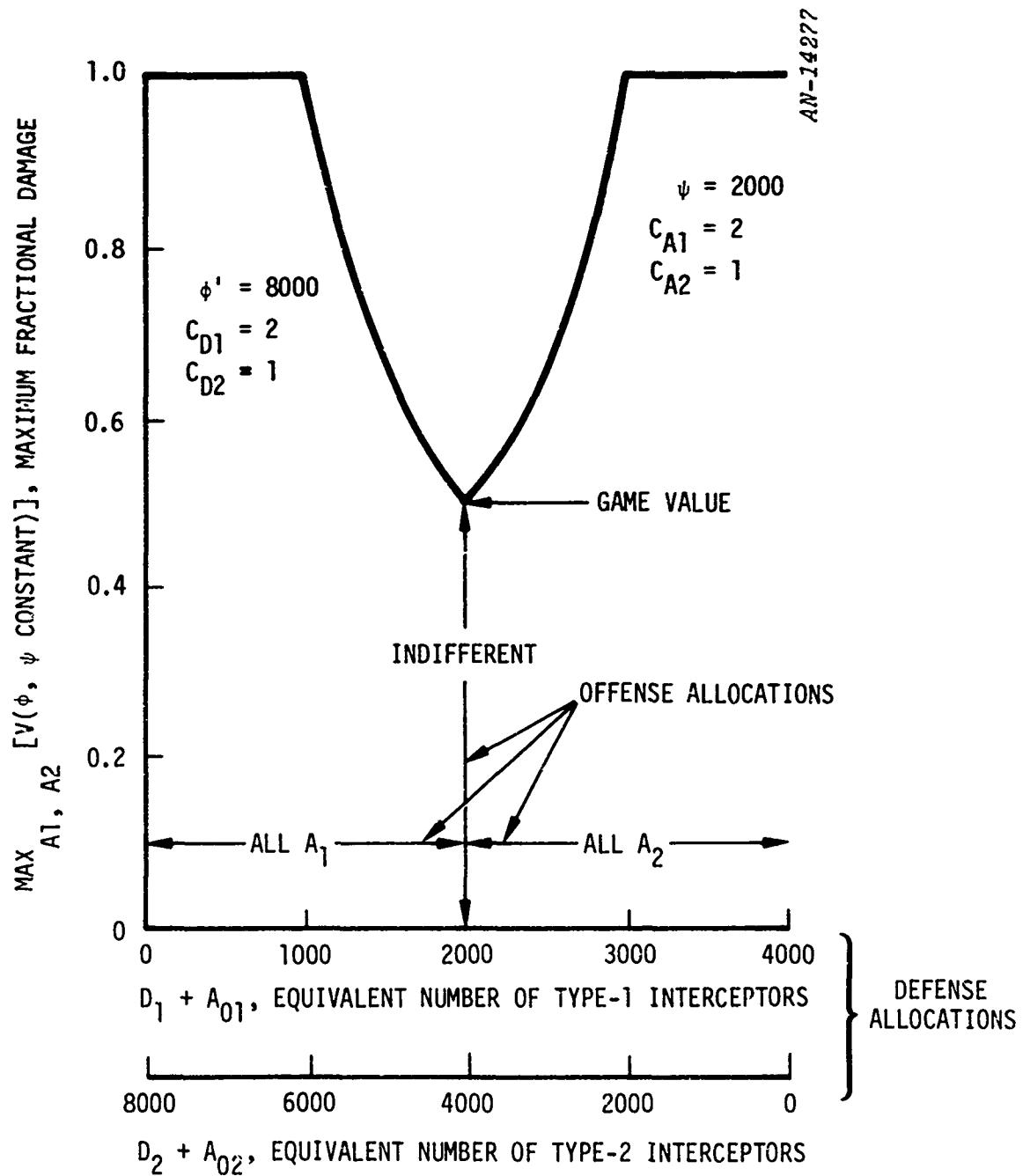


Figure 5. Example of Maximum Fractional Damage vs Defense Allocation

It is worth noting that when the defense does allocate

$$D_1 = \frac{\phi/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}, \quad D_2 = \frac{\phi/C_{A2}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}$$

V is totally indifferent to the offense allocation.

B. DEFENSE COST AS VALUE FUNCTION WITH FRACTIONAL DAMAGE AND OFFENSE RESOURCES CONSTRAINED

1. $\frac{\text{Max}_{A_1, A_2} \text{Min}_{D_1, D_2} \{\phi(V, \psi \text{ constant})\}}{\text{Max}_{A_1, A_2} \{\phi(V, \psi \text{ constant})\}}$

$$\frac{\partial \phi}{\partial D_i} + \lambda \frac{\partial V}{\partial D_i} + \mu \frac{\partial \psi}{\partial D_i} = 0, \quad i = 1, 2$$

$$C_{D1} - \lambda \frac{A_1}{D_1} = 0$$

$$C_{D2} - \lambda \frac{A_2}{D_2} = 0$$

$$V = \frac{A_1}{D_1} + \frac{A_2}{D_2}$$

These simultaneous equations solve to:

$$D_1 = \frac{A_1 + \sqrt{A_1 A_2 (C_{D2}/C_{D1})}}{V}$$

$$D_2 = \frac{A_2 + \sqrt{A_1 A_2 (C_{D1}/C_{D2})}}{V}$$

$$\text{Min}_{D_1, D_2} \{\phi(V, \psi \text{ constant})\} = \frac{(\sqrt{C_{D1} A_1} + \sqrt{C_{D2} A_2})^2}{V}$$

$$\frac{\partial}{\partial A_j} \left[\text{Min}_{D_1, D_2} \{ \phi(V, \psi \text{ constant}) \} \right] + \lambda \frac{\partial V}{\partial A_j} + \mu \frac{\partial \psi}{\partial A_j} = 0, \quad j = 1, 2$$

$$\frac{c_{D2} + \sqrt{c_{D1}c_{D2}(A_2/A_1)}}{V} + \mu c_{A1} = 0$$

$$\frac{c_{D2} + \sqrt{c_{D1}c_{D2}(A_1/A_2)}}{V} + \mu c_{A2} = 0$$

$$\psi = c_{A1}A_1 + c_{A2}A_2$$

These equations then solve to:

$$c_{A1}A_1 = \frac{c_{D1}}{\frac{c_{D1}}{c_{A1}} + \frac{c_{D2}}{c_{A2}}} \psi$$

$$c_{A2}A_2 = \frac{c_{D2}}{\frac{c_{D1}}{c_{A1}} + \frac{c_{D2}}{c_{A2}}} \psi$$

and

$$\text{Max}_{A_1, A_2} \text{Min}_{D_1, D_2} \{ \phi(V, \psi \text{ constant}) \} = \frac{\psi}{V} \left(\frac{c_{D1}}{c_{A1}} + \frac{c_{D2}}{c_{A2}} \right)$$

and the Max Min allocations of D_1 and D_2 are:

$$D_1 = \frac{\psi}{c_{A1}V}$$

$$D_2 = \frac{\psi}{c_{A2}V}$$

This result is quite interesting:

The game-value defense strategy is to allocate as if the offense could, unknown to the defense, put all of its resources into either threat. The strategy is enforceable over all possible offense resource allocations, and it can be enforced by the offense by allocating its resources in proportion to the interceptor-to-attacker cost ratio.

2. $\frac{\text{Min}_{D_1, D_2} \text{Max}_{A_1, A_2} \{\phi(V, \psi \text{ constant})\}}{}$

$$\frac{\partial \phi}{\partial A_i} + \lambda \frac{\partial V}{\partial A_i} + \mu \frac{\partial \psi}{\partial A_i} = 0 ; i = 1, 2$$

$$\left. \begin{array}{l} \frac{\lambda}{D_1} + \mu C_{A1} = 0 \\ \frac{\lambda}{D_2} + \mu C_{A2} = 0 \end{array} \right\} C_{A1} D_1 = C_{A2} D_2 = C$$

$$V = \frac{C_{A1} A_1 + C_{A2} A_2}{C} = \frac{\psi}{C} ; C = \frac{\psi}{V}$$

so that this only solves for

$$D_1 = \frac{\psi}{C_{A1} V} , D_2 = \frac{\psi}{C_{A2} V}$$

i.e., for the game-value defense strategy. In order that the defense limit the damage to V, if the offense moves last, we must have

$$D_1 \geq \frac{\psi}{C_{A1} V}$$

$$D_2 \geq \frac{\psi}{C_{A2} V}$$

So there is no reasonable physical significance to off-game-value defense strategies in this formulation (when the offense has the last move in allocation) since the only permissible off-game-value strategies are those where the defense buys more of one (or more) resource than needed to enforce the damage V against all possible offense allocations.

C. OFFENSE COST AS VALUE FUNCTION WITH FRACTIONAL DAMAGE AND DEFENSE RESOURCES CONSTRAINED

1. $\frac{\text{Min}_{A_1, A_2} \text{Max}_{D_1, D_2} \{ \psi(V, \phi \text{ constant}) \}}{}$

$$\frac{\partial \psi}{\partial D_i} + \lambda \frac{\partial V}{\partial D_i} + \mu \frac{\partial \phi}{\partial D_i} = 0 ; i = 1, 2$$

$$\frac{-\lambda A_1}{D_1^2} + \mu C_{D1} = 0$$

$$\frac{-\lambda A_2}{D_2^2} + \mu C_{D2} = 0$$

$$\phi = C_{D1} D_1 + C_{D2} D_2$$

These equations are completely equivalent to case A (fractional damage as value function) and again solve to:

$$D_1 = \frac{\phi}{\sqrt{C_{D1} C_{D2} (A_2/A_1) + C_{D1}}}$$

$$D_2 = \frac{\phi}{\sqrt{C_{D1} C_{D2} (A_1/A_2) + C_{D2}}} \quad \text{and}$$

$$V = \frac{\left(\sqrt{C_{D1} A_1} + \sqrt{C_{D2} A_2} \right)^2}{\phi}$$

$$\frac{\partial}{\partial A_j} \left[\max_{D_1, D_2} \left\{ \psi(V, \phi \text{ constant}) \right\} \right] + \lambda \frac{\partial V}{\partial A_j} + \mu \frac{\partial \phi}{\partial A_j} = 0 ; \quad j = 1, 2$$

$$C_{A1} + \frac{\lambda}{\phi} (C_{D1} + \sqrt{C_{D1} C_{D2} (A_2/A_1)}) = 0$$

$$C_{A2} + \frac{\lambda}{\phi} (C_{D2} + \sqrt{C_{D1} C_{D2} (A_1/A_2)}) = 0$$

These simultaneous equations again yield

$$\frac{A_2}{A_1} = \left(\frac{C_{A1}}{C_{A2}} \right)^2 \frac{C_{D2}}{C_{D1}}$$

Substituting in

$$V = \frac{\left(\sqrt{C_{D1} A_1} + \sqrt{C_{D2} A_2} \right)^2}{\phi} = A_1 \frac{\left[\sqrt{C_{D1}} + \sqrt{C_{D2} (A_2/A_1)} \right]^2}{\phi}$$

yields

$$C_{A1} A_1 = \frac{C_{D1}/C_{A1}}{\left[(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) \right]^2} V \phi$$

Similarly,

$$C_{A2} A_2 = \frac{C_{D2}/C_{A2}}{\left[(C_{D1}/C_{A1}) + (C_{D2}/C_{A2}) \right]^2} V \phi$$

and

$$\min_{A_1, A_2} \max_{D_1, D_2} \{ \psi (V, \phi \text{ constant}) \} = \frac{V \phi}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}$$

$$2. \quad \frac{\text{Max}_{D_1, D_2} \text{Min}_{A_1, A_2} \{ \psi (V, \phi \text{ constant}) \}}{}$$

$$\frac{\partial \psi}{\partial A_j} + \lambda \frac{\partial V}{\partial A_j} + \mu \frac{\partial \phi}{\partial A_j} = 0 ; \quad j = 1, 2$$

$$c_{A1} + \frac{\lambda}{D_1} = 0$$

$$c_{A2} + \frac{\lambda}{D_3} = 0$$

$$\phi = c_{D1} D_1 + c_{D3} D_3$$

These are the same equations as in case A (fractional damage as value function); therefore, the games are equivalent and for off-game-value defense allocations:

$$D_1 \leq \frac{\phi/c_{A1}}{(c_{D1}/c_{A1}) + (c_{D2}/c_{A2})}$$

$$A_1 = VD_1 \quad \& \quad A_2 = 0, \quad \therefore$$

$$\psi = D_1 V c_{A1}$$

$$D_1 \geq \frac{\phi/c_{A1}}{(c_{D1}/c_{A1}) + (c_{D2}/c_{A2})}$$

$$A_2 = VD_2 = \frac{V}{c_{D2}} (\phi - c_{D1} D_1) \quad \therefore$$

$$\psi = \frac{V c_{A2}}{c_{D2}} (\phi - c_{D1} D_1)$$

Figure 6 shows an example of minimized offense cost (for a fractional damage of fifty percent) vs defense allocation for:

$$V = 0.50$$

$$\phi' = 8000$$

$$c_{D1} = 2$$

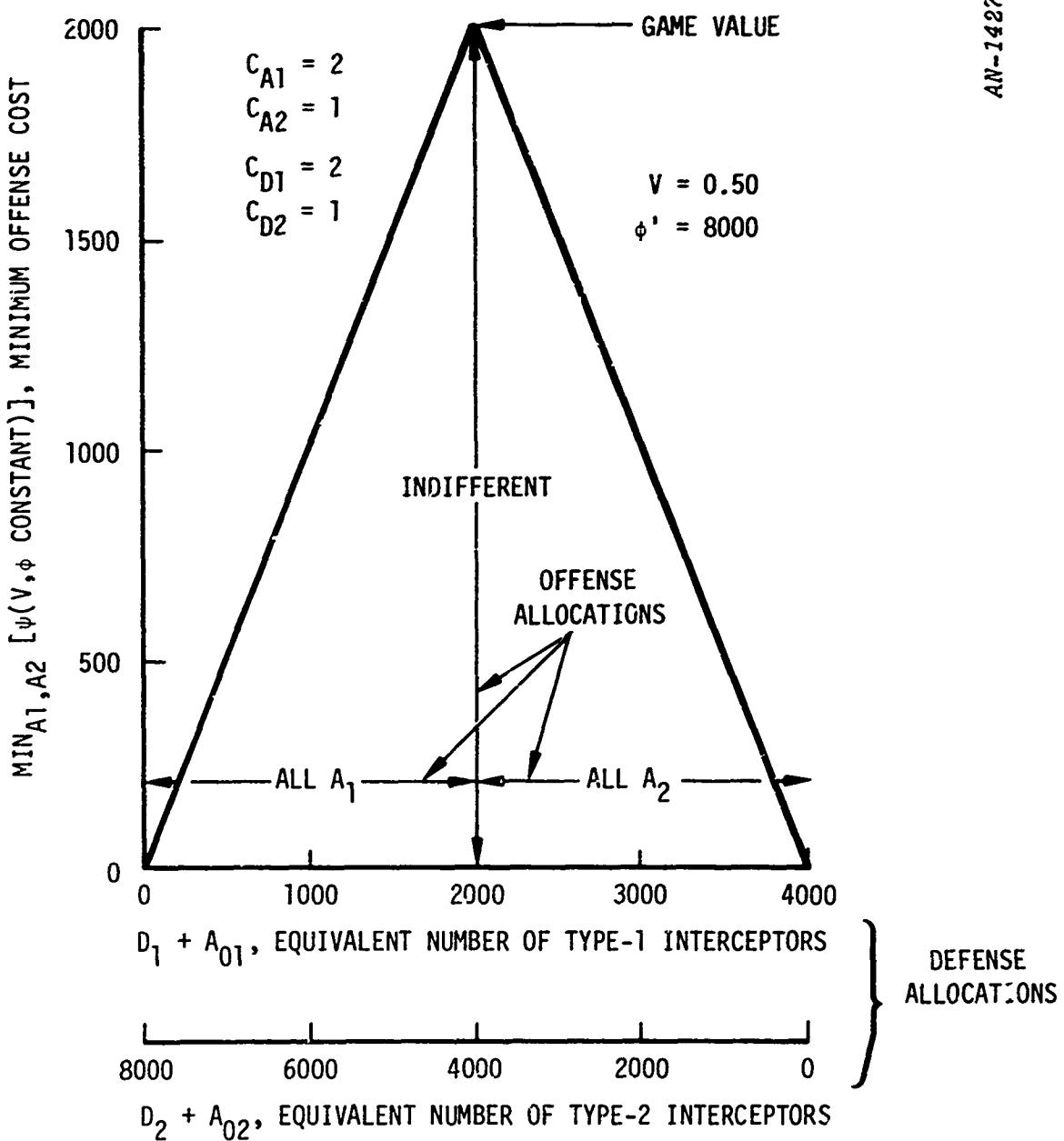


Figure 6. Example of Minimum Offense Cost vs Defense Allocation

$$C_{D2} = 1$$

$$C_{A1} = 2$$

$$C_{A2} = 1$$

Again, when the defense allocates

$$D_1 = \frac{\phi/C_{A1}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}, \quad D_2 = \frac{\phi/C_{A2}}{(C_{D1}/C_{A1}) + (C_{D2}/C_{A2})}$$

the outcome (ψ in this case) is indifferent to the offense allocation.

Table 2 summarizes the results for two-on-two games.

TABLE 2
SUMMARY OF TWO-ON-TWO GAMES

MAXIMUM DAMAGE = $V = \frac{A_1}{D_1 + A_{11}} + \frac{A_2}{D_2 + A_{22}}$	$D_1 =$ INVENTORY OF DEFENDER TYPE 1 $D_2 =$ INVENTORY OF DEFENDER TYPE 2 $A_1 =$ INVENTORY OF ATTACKER TYPE 1 $A_2 =$ INVENTORY OF ATTACKER TYPE 2 $A_{11} =$ NUMBER OF TYPE 1 TO KILL UNDEFENDED TARGET SYSTEM $A_{22} =$ NUMBER OF TYPE 2 TO KILL UNDEFENDED TARGET SYSTEM $C_{11} =$ UNIT COSTS
--	---

		CASE A $V =$ VALUE FUNCTION Max_{D_1, D_2}	CASE B $V =$ VALUE FUNCTION Max_{D_1, D_2}	CASE C $V =$ VALUE FUNCTION Max_{D_1, D_2}
Optimal Allocations for D_1 and D_2 Against Arbitrary A_1 and A_2	$D_1 = A_{11} + \sqrt{\frac{A_1}{C_{11}D_1 + C_{11}A_{11}}} + D_2 + A_{22} + \sqrt{\frac{A_2}{C_{22}D_2 + C_{22}A_{22}}}$	$D_1 = A_{11} + \sqrt{\frac{C_{12}}{A_1D_1 + C_{11}A_{11}}} + D_2 + A_{22} + \sqrt{\frac{C_{22}}{A_2D_2 + C_{22}A_{22}}}$	$D_1 = A_{11} + \sqrt{\frac{C_{12}}{A_1D_1 + C_{11}A_{11}}} + D_2 + A_{22} + \sqrt{\frac{C_{22}}{A_2D_2 + C_{22}A_{22}}}$	$D_1 = A_{11} + \sqrt{\frac{C_{12}}{A_1D_1 + C_{11}A_{11}}} + D_2 + A_{22} + \sqrt{\frac{C_{22}}{A_2D_2 + C_{22}A_{22}}}$
Corresponding Value Function	$\text{Max}_{D_1, D_2} (V) = \frac{(\sqrt{C_{11}A_1} + \sqrt{C_{22}A_2})^2}{V}$			
Game-Value Allocations for A_1 and A_2	$A_1 = \frac{C_{11}/A_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} A_{11} + A_2 = \frac{C_{22}/A_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} A_{22}$	$A_1 = \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} A_{11} + A_2 = \frac{C_{22}/C_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} A_{22}$	$A_1 = \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} A_{11} + A_2 = \frac{C_{22}/C_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} A_{22}$	$A_1 = \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} A_{11} + A_2 = \frac{C_{22}/C_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} A_{22}$
Game-Value Allocations for D_1 and D_2	$D_1 = A_{11} + \frac{C_{11}/A_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + D_2 + A_{22} = \frac{C_{22}/A_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{22} + D_{11}$	$D_1 = A_{11} + \frac{C_{11}/A_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + D_2 + A_{22} = \frac{C_{22}/A_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{22} + D_{11}$	$D_1 = A_{11} + \frac{C_{11}/A_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + D_2 + A_{22} = \frac{C_{22}/A_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{22} + D_{11}$	$D_1 = A_{11} + \frac{C_{11}/A_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + D_2 + A_{22} = \frac{C_{22}/A_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{22} + D_{11}$
Game-Value Function	$\text{Max}_{A_1, A_2} \text{Min}_{D_1, D_2} (V) = \text{Max}_{D_1, D_2} \text{Max}_{A_1, A_2} (V) = \left(\frac{C_{11}}{C_{11}} + \frac{C_{22}}{C_{22}} \right) \frac{1}{2}$	$\text{Max}_{A_1, A_2} \text{Min}_{D_1, D_2} (V) = \text{Max}_{D_1, D_2} \text{Max}_{A_1, A_2} (V) = \left(\frac{C_{11}}{C_{11}} + \frac{C_{22}}{C_{22}} \right) \frac{1}{2}$	$\text{Max}_{A_1, A_2} \text{Min}_{D_1, D_2} (V) = \text{Max}_{D_1, D_2} \text{Max}_{A_1, A_2} (V) = \left(\frac{C_{11}}{C_{11}} + \frac{C_{22}}{C_{22}} \right) \frac{1}{2}$	$\text{Max}_{A_1, A_2} \text{Min}_{D_1, D_2} (V) = \text{Max}_{D_1, D_2} \text{Max}_{A_1, A_2} (V) = \left(\frac{C_{11}}{C_{11}} + \frac{C_{22}}{C_{22}} \right) \frac{1}{2}$
Optimal Allocations for A_1 and A_2 Against off-optimal D_1 and D_2	$D_1 = A_{11} + \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + D_2 = \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + \frac{C_{22}/A_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{22}$	$D_1 = A_{11} + \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + D_2 = \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + \frac{C_{22}/A_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{22}$	$D_1 = A_{11} + \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + D_2 = \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + \frac{C_{22}/A_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{22}$	$D_1 = A_{11} + \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + D_2 = \frac{C_{11}/C_{11}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{11} + \frac{C_{22}/A_{22}}{(C_{11}/A_{11})^2 + (C_{22}/A_{22})^2} D_{22}$
Corresponding Value Function	$V = \frac{C_{11}A_1}{C_{11}A_1 + C_{22}A_2}$			

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13. ABSTRACT A simple game-theoretic representation of terminal defense against multicomponent offense has been investigated for two cases: 1. Each offense component must be countered by a "threat-specific" interceptor type. For example, reentry vehicles (RVs) are countered by an ABM interceptor such as SPRINT, high-altitude air-to-surface missiles (ASMs) by a high-altitude missile such as Hercules, and low-altitude ASMs by a low-altitude missile such as HAWK. 2. Some interceptors are multipurpose. For example, suppose SPRINT could intercept RVs and high- and low-altitude ASMs, Hercules could intercept high- and low-altitude ASMs, and HAWK could intercept only low-altitude ASMs. With single-purpose interceptors there is a (game value) offense allocation that forces the defense to respond as if the offense (unknown to the defense) allocated all of its resources to any one of the offense threat types. The outcome, with this defense response, is completely independent of the actual offense allocation (i.e., it is defense-enforceable). Most reasonable offense allocations require essentially the same defense response, since the allocation optimization is "flat" with respect to defense resources but "spiked" (discontinuous) with respect to defense resources.		

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13. ABSTRACT (Continued) With multipurpose interceptors, the same statements are true. But the effective number of interceptors available against any threat type is the sum of the numbers of all interceptor types effective against that threat. The result is that the offense's advantage in deploying more than one component is strongly reduced, or even reversed. Several examples are worked out in this paper.		

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